

# The Density of Fan-Planar Graphs

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## Abstract

A topological drawing of a graph is fan-planar if for each edge  $e$  the edges crossing  $e$  have a common endpoint on the same side of  $e$ , and a fan-planar graph is a graph admitting such a drawing. Equivalently, this can be formulated by two forbidden patterns, one of which is the configuration where  $e$  is crossed by two independent edges and the other where  $e$  is crossed by incident edges with the common endpoint on different sides of  $e$ . In particular every edge of a fan-planar graph is crossed only by the edges of a star. A topological drawing is simple if any two edges have at most one point in common.

The class of fan-planar graphs is a natural variant of other classes defined by forbidden intersection patterns in a topological drawing of the graph. So every 1-planar graph is also fan-planar, and every fan-planar graph is also quasipolar, where both inclusions are strict. Fan-planar graphs also fit perfectly in a recent series of work on nearly-planar graphs from the area of graph drawing and combinatorial embeddings.

For topologically defined graph classes, one of the most fundamental questions asks for the maximum number of edges in any such graph with  $n$  vertices. We prove that every  $n$ -vertex graph without loops and parallel edges that admits a simple fan-planar drawing has at most  $5n - 10$  edges and that this bound is tight for every  $n \geq 20$ .

Furthermore we discuss possible extensions and generalizations of these new concepts.

*Keywords:* Topological drawing, quasipolar, 1-planar, intersection pattern, density.

## 1 Introduction

Planarity of a graph is a well-studied concept in graph theory, computational geometry and graph drawing. The famous Euler formula characterizes for a certain embedding the relation between vertices, edges and faces, and many different algorithms [28, 23, 11] following different objectives have been developed to compute appropriate embeddings in the plane.

Because of the importance of the concepts, a series of generalizations have been developed in the past. Topological graphs and topological drawings respectively are being considered, i.e., the vertices are drawn as points in the plane and the edges drawn as Jordan curves between corresponding points without any other vertex as an interior point. In [16], the authors state "Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory" together with 9 citations from a large group of authors. Most of the existent literature considers topological drawings that are *simple*, i.e., where any two edges have at most one point in common. In particular, two edges may not cross more than once and incident edges may not cross at all. Throughout this paper we shall consider simple topological graphs only. Indeed, we shall argue in Section 4 that if we drop this assumptions and allow non-homeomorphic parallel edges, then even 3-vertex fan-planar graphs have arbitrarily many edges.

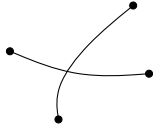
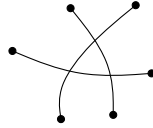
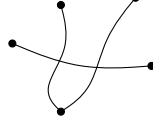

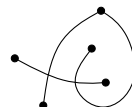
| Topological Graphs Defined by Forbidden Intersection Patterns                     |   |   |   |
|---|---|---|---|
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| planar<br>$\leq 3n - 6$ edges   | 3-quasiplanar planar<br>$\leq 6.5 \cdot n + O(1)$ edges                           | 2-fan-crossing free<br>$\leq 4n - 8$ edges  | fan-planar<br>$\leq 5n - 10$ edges  |

Figure 1: Topological graphs defined by forbidden patterns and the corresponding maximum number of edges in an  $n$ -vertex such graph.

**Related work.** Most notably are the  $k$ -planar graphs and the  $k$ -quasiplanar graphs [4]. A  $k$ -planar graph admits a topological drawing in which no edge is crossed more than  $k$  times by other edges, while a  $k$ -quasiplanar graph admits a drawing in which no  $k$  edges pairwise cross each other.

The topic of  $k$ -quasiplanar graphs is almost classical [9]. A famous conjecture [9] states that for constant  $k$  the maximal number of edges in  $k$ -quasiplanar graphs is linear in the number of vertices. Note that 2-quasiplanar graphs correspond to planar graphs. A first linear bound for  $k = 3$ , i.e. 3-quasiplanar graphs, has been shown in [4] and subsequently improved in [21]. For 4-quasiplanar graphs the current best bound is  $76(n-2)$  [1]. For the general case, the bounds have been gradually improved from  $O(n(\log n)^{O(\log k)})$  [21], and  $O(n \log n \cdot 2^{\alpha(n)^c})$ .

In case of simple topological drawings, where each pair of edges intersects at most once, a bound of  $6.5n + O(1)$  has been proven for 3-quasiplanar graphs [3] and recently  $O(n \log n)$  for  $k$ -quasiplanar graphs with any fixed  $k \geq 2$  [24]. It is still open, if the conjecture holds for general  $k$ .

A  $k$ -planar graph admits a topological drawing in which each edge has at most  $k$  crossings. The special case of 1-planar graphs have been introduced by Ringel [22], who considered the chromatic number of these graphs. Important work about the characterization on 1-planar graphs has been performed by Suzuki [25], Thomassen [27] and Hong *et al.* [19]. Related questions on testing 1-planarity have been explored, where NP-completeness has been shown for the general case [17] while efficient algorithms have been found for testing 1-planarity for a given rotation system [14] and for the case of outer-planarity [7, 18]. Additionally aspects like straight-line embeddings [5] and maximality [8] etc. have been explored in the past.

Closely related to 1-planar graphs are RAC-drawable graphs [13, 6], that is graphs that can be drawn in the plane with straight-line edges and right-angle crossings. For the maximum number of edges in such a graph with  $n$  vertices, a bound of  $4n - 10$  could be proven [15], which is remarkably close to the  $4n - 8$  bound for the class of 1-planar graphs [21]. A necessary condition for RAC-drawable graph is the absence of fan-crossings. An edge has a  $k$ -fan-crossing if it crosses  $k$  edges that have a common endpoint, cf. Figure 1. RAC-drawings do not allow 2-fan-crossings. In a recent paper [10], Cheong *et al.* considered  $k$ -fan-crossing free graphs and gave bounds for their maximum number of edges. They obtain a tight bound of  $4n - 8$  for  $n$ -vertex 2-fan-crossing free graphs, and a tight  $4n - 9$  when edges are required to be straight-line segments. For  $k > 2$ , they prove an upper bound of  $3(k-1)(n-2)$  edges, while all known examples of  $k$ -fan-crossing free graphs on  $n$  vertices have no more than  $kn$  edges.

**Our results and more related work.** Throughout this paper we consider only simple topological drawings, i.e., any two edges have at most one point in common, and only simple graphs, i.e., graphs without loops and parallel edges. We consider here another variant of sparse non-planar graphs,

somehow halfway between 1-planar graphs and quasiplanar graphs, where we allow more than one crossing on an edge  $e$ , but only if the crossing edges have a common endpoint on the same side of  $e$ . We call this a **fan-crossing** and the class of topological graphs obtained this way **fan-planar graphs**. Note that we do not differentiate on  $k$ -fan-crossings as it has been done by Cheong *et al.* [10].

The requirement that every edge in  $G$  is crossed by a fan-crossing can be stated in terms of forbidden configurations. We define *configuration I* to be one edge that is crossed by two independent edges, and *configuration II* to be an edge  $e$  that is crossed by incident edges, which however have their common endpoint on different sides of  $e$ , see Figure 2. Note that since we consider only simple topological drawings, configuration II is well-defined. Now a simple topological graph is fan-planar if and only if neither configuration I nor II occurs. Note that if we forbid only configuration I, then an edge may be crossed by the three edges of a triangle, which is actually not a star, nor a fan-crossing. However, if every edge is drawn as a straight-line segment, then configuration II can not occur and hence in this case it is enough to forbid configuration I.

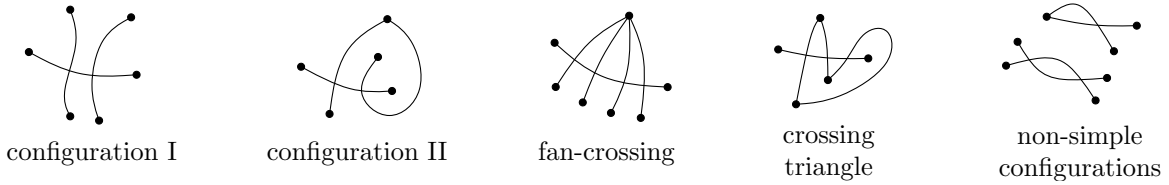


Figure 2: Crossing configurations

Obviously, 1-planar graphs are also fan-planar. Furthermore, fan-planar graphs are 3-quasiplanar since there are no three independent edges that mutually cross. So, we know already that the maximum number of edges in an  $n$ -vertex fan-planar graph is approximately between  $4n$  and  $6.5n$ . In the following, we will explore the exact bound.

**Theorem 1.** *Every simple topological graph  $G$  on  $n \geq 3$  vertices with neither configuration I nor configuration II has at most  $5n - 10$  edges. This bound is tight for  $n \geq 20$ .*

We remark that fan-planar drawings graphs may have  $\Omega(n^2)$  crossings, e.g., a straight-line drawing of  $K_{2,n}$  with the bipartition classes placed on two parallel lines.

Very closely related to our approach is the research on forbidden grids in topological graphs, where a  $(k, l)$  grid denotes a  $k$ -subset of the edges pairwise intersected by an  $l$ -subset of the edges, see [20] and [26]. It is known that topological graphs without  $(k, l)$  grids have a linear number of edges if  $k$  and  $l$  are fixed. Note that configuration I, but also a 2-fan-crossing, are  $(2, 1)$  grids. Subsequently [2], "natural"  $(k, l)$  grids have been considered, which have the additional requirement that the  $k$  edges, as well as the  $l$  edges, forming the grid are pairwise disjoint. For natural grids, the achieved bounds are superlinear. Linear bounds on the number of edges have been found for the special case of forbidden natural  $(k, 1)$  grids where the leading constant heavily depends on the parameter  $k$ . In particular, the authors give a bound of  $65n$  for the case of forbidden natural  $(2, 1)$  grids, which correspond to our forbidden configuration I. Additionally, the case of geometric graphs, that is, graphs with straight-line edges, has been explored. For details and differences let us refer to [2]. We remark that many arguments in this field of research are based on the probabilistic method, while in this paper we use a direct approach aiming on tight upper and lower bounds.

## 2 Examples of Fan-Planar Graphs with Many Edges

The following examples have approximately  $5n$  edges. The first one is a  $K_{4,n-4}$ , where the  $n-4$  edges are connected by a path, see Figure 3(a). An easy calculation shows that this graph has  $4(n-4) + (n-5) = 5n - 21$  edges. Indeed, one can add 10 edges to the graph, keeping fan-planarity, as well as additionally one vertex with 6 more incident edges and obtain a graph on  $n+1$  vertices and  $5(n+1) - 10$  edges. We remark that this graph has parallel edges; however every pair of parallel edges is non-homeomorphic, that is, it surrounds at least one vertex of  $G$ .

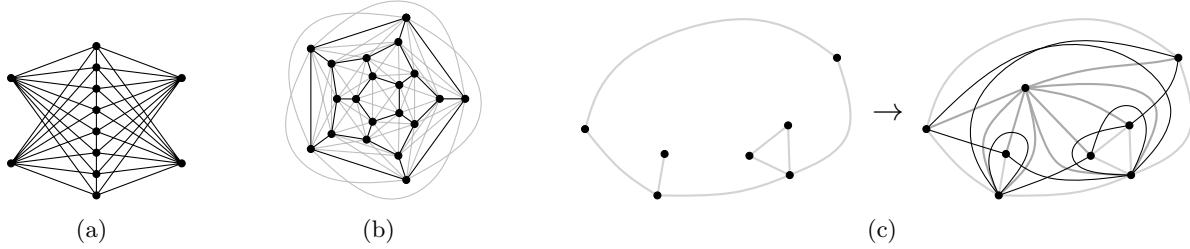


Figure 3: (a)  $K_{4,n-4}$  with  $n-4$  vertices on a path. (b) The dodecahedral graph with a pentagram in each face. (c) Adding 2-hops and spokes into a face.

The second example is the (planar) dodecahedral graph where in each 5-face, we draw 5 additional edges as a pentagram, see Figure 3(b). This graph has  $n = 20$  vertices and  $5n - 10 = 90$  edges, and has already served as a tight example for 2-planar graphs [21].

**Proposition 1.** *Every connected planar embedded graph  $H$  on  $n \geq 3$  vertices can be extended to a fan-planar graph  $G$  with  $5|V(G)| - 10$  edges by adding an independent set of vertices and sufficiently many edges, such that the uncrossed edges of  $G$  are precisely the edges of  $H$ .*

*Moreover, if  $H$  is 3-connected and each face has length at least 5, then  $G$  is a simple topological graph without loops or parallel edges.*

*Proof.* Let  $n$  and  $m$  be the number of vertices and edges of  $H$ , respectively, and  $F$  be the set of all faces of  $H$ . We construct the fan-planar graph  $G$  by adding one vertex and two sets of edges into each face  $f \in F$ . So let  $f$  be any face of  $H$ . Since  $H$  is connected,  $f$  corresponds to a single closed walk  $v_1, \dots, v_s$  in  $H$  around  $f$ , where vertices and edges may be repeated. We do the following, which is illustrated in Figure 3(c).

- (1) Add a new vertex  $v_f$  into  $f$ .
- (2) For  $i = 1, \dots, s$  add a new edge  $v_f v_i$  drawn in the interior of  $f$ .
- (3) For  $i = 1, \dots, s$  add a new edge  $v_{i-1} v_{i+1}$  (with indices modulo  $s$ ) crossing the edge  $v_f v_i$ .

In (1) we added  $|F|$  new vertices. In (2) we added  $\deg(f)$  many “spoke edges” inside face  $f$ , in total  $\sum_f \deg(f) = 2m$  new edges. And in (3) we added again  $\deg(f)$  many “2-hop edges” inside face  $f$ , in total  $\sum_f \deg(f) = 2m$  new edges. Thus we calculate

$$\begin{aligned} |V(G)| &= n + |F| \\ |E(G)| &= m + 2m + 2m = 5m, \end{aligned}$$

which together with Euler’s formula  $m = n + |F| - 2$  gives  $|E(G)| = 5|V(G)| - 10$ . It remains to see that no two edges in  $G$  are homeomorphic, and that  $G$  is fan-planar. The “2-hop edges” form shortcuts for paths of length 2. Since  $s \geq 4$  by assumption, none of these  $s$  edges is already

in the facial walk for  $f$ . Each “spoke edge”  $v_f v_i$  crosses only one 2-hop edge, and each 2-hop edge  $v_{i-1} v_{i+1}$  crosses only three edges  $v_{i-2} v_i$ ,  $v_f v_i$  and  $v_i v_{i+2}$ , which have  $v_i$  as a common endpoint. Hence the resulting graph  $G$  is fan-planar.

Finally, note that if the planar graph  $H$  is 3-connected and each face has length at least 5, then the fan-planar graph  $G$  has no loops, nor parallel edges, nor crossing incident edges. Examples for such planar graphs are fullerene graphs.  $\square$

### 3 The $5n - 10$ Upper Bound For the Number of Edges

In this section we prove Theorem 1. We shall fix a fan-planar embedding of  $G$  and split the edges of  $G$  into three sets. The first set contains all edges that are uncrossed. We denote by  $H$  the subgraph of  $G$  with all vertices in  $V$  and all uncrossed edges of  $G$ . Sometimes we may refer to  $H$  as the *planar subgraph of  $G$* . Note that  $H$  might be disconnected even if  $G$  is connected. In the second set we consider every crossed edge whose endpoints lie in the same connected component of  $H$ . And the third set contains all remaining edges, i.e., every crossed edge with endpoints in different components of  $H$ . We show how to count the edges in each of the three sets and derive the upper bound.

To prove Theorem 1 it clearly suffices to consider simple topological graphs  $G$  that do not contain configuration I nor II and additionally satisfy the following properties.

- (i) The chosen embedding of  $G$  has the maximum number of uncrossed edges.
- (ii) The addition of any edge to the given embedding violates the fan-planarity of  $G$ , that is,  $G$  is maximal fan-planar with respect to the given embedding.

So for the remainder of this paper let  $G$  be a maximal fan-planar graph with a fixed fan-planar embedding that has the maximum number of uncrossed edges. Recall that the embedding of  $G$  is simple, i.e., any two edges have at most one point in common.

#### 3.1 Notation, Definitions and Preliminaries Results

We call a connected component of the plane after the removal of all vertices and edges of  $G$  a *cell of  $G$* . Whenever we consider a subgraph of  $G$  we consider it together with its fan-planar embedding, which is inherited from the embedding of  $G$ . We will sometimes consider cells of a subgraph  $G'$  of  $G$ , even though those might contain vertices and edges of  $G - G'$ . The boundary of each cell  $c$  is composed of a number of edge segments and some (possibly none) vertices of  $G'$ . With slight abuse of notation we call the cyclic order of vertices and edge segments along  $c$  the *boundary of  $c$* , denoted by  $\partial c$ . Note that vertices and edges may appear more than once in the boundary of a single cell. We define the *size of a cell  $c$* , denoted by  $||c||$ , as the total number of vertices and edge segments in  $\partial c$  counted with multiplicity.

Note that from the additional assumptions (i) and (ii) on  $G$  it follows that if two vertices are in the same cell  $c$  of  $G$  then they are connected by an uncrossed edge of  $G$ . However, this uncrossed edge does not necessarily bound the cell  $c$ .

**Lemma 1.** *If two edges  $vw$  and  $ux$  cross in a point  $p$ , no edge at  $v$  crosses  $ux$  between  $p$  and  $u$ , and no edge at  $x$  crosses  $vw$  between  $p$  and  $w$ , then  $u$  and  $w$  are contained in the same cell of  $G$ .*

*Proof.* Let  $e_0 = ux$  and  $e_1 = vw$  be two edges that cross in point  $p = p_1$  such that no edge at  $v$  crosses  $e_0$  between  $p_1$  and  $u$ , and no edge at  $x$  crosses  $e_1$  between  $p_1$  and  $w$ . If no edge of  $G$  crosses  $e_0$  nor  $e_1$  between  $p_1$  and  $u$ , respectively  $w$ , then clearly  $u$  and  $w$  are bounding the same

cell. So assume without loss of generality that some edge of  $G$  crosses  $e_1$  between  $p_1$  and  $w$ . By fan-planarity such edges are incident to  $u$ . Let  $e_2$  be the edge whose crossings with  $e_1$  is closest to  $w$ , and let  $p_2$  be the crossing point. See Figure 4(a) for an illustration.

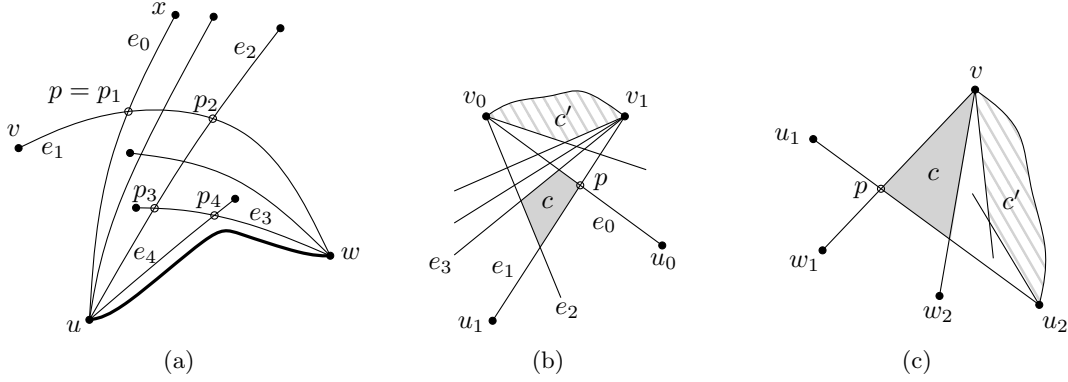


Figure 4: Illustration of the proofs of Lemma 1 (a) and Corollary 2 (b),(c).

No edge crosses  $e_1$  between  $w$  and  $p_2$ . If  $e_2$  is not crossed between  $u$  and  $p_2$ , then  $u$  and  $w$  are bounding the same cell and we are done. Otherwise let  $e_3$  be the edge whose crossing with  $e_2$  is closest to  $u$ , and let  $p_3$  be the crossing point. By fan-planarity  $e_3$  and  $e_1$  have a common endpoint, and it is not  $v$  since  $e_3$  does not cross  $e_0$  between  $p_1$  and  $u$ . So  $e_3$  endpoints at  $w$  and we have that  $e_2$  is not crossed between  $u$  and  $p_3$ . Again, if  $u$  and  $w$  are not on the same cell then some edge crosses  $e_3$  between  $p_3$  and  $w$ . By fan-planarity any such edge has a common endpoint with  $e_2$ , and if it would not be  $u$  then  $e_1$  would be crossed by two independent edges – a contradiction to the fan-planarity of  $G$ . So all edges crossing  $e_3$  between  $w$  and  $p_3$  are incident to  $u$ . Let  $e_4$  be such edge whose crossing with  $e_3$  is closest to  $w$ , and let  $p_4$  be the crossing point. Let us again refer to Figure 4(a) for an illustration.

Iterating this procedure until no edge crosses  $e_i$  nor  $e_{i-1}$  between  $p_i$  and  $u, w$  we see that  $u$  and  $w$  lie indeed on the same cell, which concludes the proof.  $\square$

Lemma 1 has a couple of nice consequences.

**Corollary 1.** *Any two crossing edges in  $G$  are connected by an uncrossed edge.*

*Proof.* Let  $ux$  and  $vw$  be the two crossing edges. By fan-planarity either no other edge at  $x$  or no other edge at  $u$  crosses the edge  $vw$ , say there is no such edge at  $x$ . Similarly, we may assume without loss of generality, that no edge at  $v$  crosses the edge  $ux$ . However, this implies that  $ux$  and  $vw$  satisfy the requirements of Lemma 1 and we have that  $u$  and  $w$  are on the same cell. In particular, we can draw an uncrossed edge between  $u$  and  $w$  in this cell. Because  $G$  is maximally fan-planar,  $uw$  is indeed an edge of  $G$ . And since  $G$  is embedded with the maximum number of uncrossed edges,  $uw$  is also drawn uncrossed.  $\square$

**Corollary 2.** *If  $c$  is a cell of any subgraph of  $G$ , and  $||c|| = 4$ , then  $c$  contains no vertex of  $G$  in its interior.*

*Proof.* Let  $c$  be a cell of  $G' \subseteq G$  with  $||c|| = 4$ . Then  $\partial c$  consists either of four edge segments or one vertex and three edge segments. Let us assume for the sake of contradiction that  $c$  contains a set  $S \neq \emptyset$  of vertices in its interior.

*Case 1.*  $\partial c$  consists of four edge segments. Let  $e_0, e_1, e_2, e_3$  be the edges bounding  $c$  in this cyclic order. From the fan-planarity of  $G$  follows that  $e_0$  and  $e_2$  have a common endpoint  $v_0$ . Similarly

$e_1$  and  $e_3$  have a common endpoint  $v_1$ . See Figure 4(b) for an illustration. If  $p$  denotes the crossing point of  $e_0 = v_0u_0$  and  $e_1 = v_1u_1$ , then by fan-planarity no edge at  $u_i$  crosses  $e_{i+1}$  between  $p$  and  $v_{i+1}$ , where  $i \in \{0, 1\}$  and indices are taken modulo 2. Hence by Lemma 1 there exists a cell  $c'$  of  $G$  that contains both  $v_0$  and  $v_1$ .

Now consider the subgraph  $G[S]$  of  $G$  on the vertices inside  $c$ . From the fan-planarity follows that every edge between  $G[S]$  and  $G[V \setminus S]$  has as one endpoint  $v_0$  or  $v_1$ . We now change the embedding of  $G$  by placing the subgraph  $G[S]$  (keeping its inherited embedding) into the cell  $c'$  that contains  $v_0$  and  $v_1$ . The resulting embedding of  $G$  is still fan-planar and moreover at least one edge between  $G[S]$  and  $\{v_0, v_1\}$  is now uncrossed – a contradiction to our assumption (i) that the embedding of  $G$  has the maximum number of uncrossed edges.

*Case 2.  $\partial c$  consists of one vertex and three edge segments.* Let  $v$  be the vertex and  $vw_1, vw_2, u_1u_2$  be the edges bounding  $c$ . See Figure 4(c) for an illustration. If  $p$  denotes the crossing point of  $vw_1$  and  $u_1u_2$ , then by fan-planarity either no edge at  $u_1$  crosses  $vw_1$  between  $p$  and  $v$  or no edge at  $u_2$  crosses  $vw_1$  between  $p$  and  $v$ . Moreover, for  $i = 1, 2$  the edge  $vw_i$  is the only edge at  $w_i$  that crosses  $u_1u_2$ . Hence by Lemma 1 we have that either  $v$  and  $u_1$  or  $v$  and  $u_2$  are contained in the same cell of  $G$  – say cell  $c'$  contains  $v$  and  $u_2$ .

Now, similarly to the previous case, consider the subgraph  $G[S]$  of  $G$  on the vertices inside  $c$ . From the fan-planarity, it follows that every edge between  $G[S]$  and  $G[V \setminus S]$  has as one endpoint  $v, u_1$  or  $u_2$ . Moreover, every edge between a vertex in  $G[S]$  and  $u_1$  or  $u_2$  is crossed only by edges incident to  $v$ , as otherwise  $u_1u_2$  would be crossed by two independent edges. We now change the embedding of  $G$  by placing the subgraph  $G[S]$  (keeping its inherited embedding) into the cell  $c'$  that contains  $v$  and  $u_2$ . The resulting embedding of  $G$  is still fan-planar and moreover at least one edge between  $G[S]$  and  $u_2$  is now uncrossed – a contradiction to (i).  $\square$

**Corollary 3.** *If  $e_0 = u_0v_0$  and  $e_1 = u_1v_1$  are two crossing edges of  $G$  such that every edge of  $G$  crossing  $e_i$  is crossed only by edges incident to  $u_{i+1}$ , where  $i \in \{0, 1\}$  and indices are taken modulo 2, then  $v_0$  and  $v_1$  are in the same connected component of  $H$ .*

*Proof.* Let  $p$  be the point in which  $e_0$  and  $e_1$  cross. For  $i = 0, 1$  let  $S_i$  be the set of all edges crossing  $e_{i+1}$  between  $p$  and  $v_{i+1}$ . (All indices are taken modulo 2.) By assumption  $S_i$  is a star centered at  $u_i$ . Consider the embedding of the graph  $S_0 \cup S_1$  inherited from  $G$ . By fan-planarity  $u_0$  and  $u_1$  are contained in the outer cell of  $S_0 \cup S_1$ . Moreover, every inner cell  $c$  of  $S_0 \cup S_1$  has  $||c|| = 4$  and thus by Corollary 2 all leaves of  $S_0$  and  $S_1$  are also contained in the outer cell  $c^*$  of  $S_0 \cup S_1$ .

We claim that no edge segment in the boundary  $\partial c^*$  of the outer cell is crossed by another edge in  $G$ . Indeed, if  $e'$  is an edge crossing some edge  $e \in S_0 \cup S_1$  between the crossing of  $e$  and  $e_0$  or  $e_1$  and the endpoint of  $e$  different from  $u_0, u_1$ , then by assumption one endpoint of  $e'$  is  $u_0$  or  $u_1$  – say  $u_1$ . Moreover, since by Corollary 2 no cell  $c$  with  $||c|| = 4$  contains any vertex, we have that  $e'$  crosses  $e_0$  between  $p$  and  $v_0$  and thus  $e \in S_1$ . See Figure 5(b).

We conclude that if we label the vertices of  $S_0 \cup S_1$  such that their cyclic order around  $c^*$  is  $u_0, u_1, v_0 = w_1, w_2, \dots, w_k = v_1$ , then for each  $j \in \{1, \dots, k-1\}$  the vertices  $w_j$  and  $w_{j+1}$  are contained in the same cell of  $G$  and hence by maximality of  $G$  joint by an uncrossed edge. See Figure 5(a) for an illustration.  $\square$

Recall that  $H$  denotes the planar subgraph of  $G$ . For convenience we refer to the closure of cells of  $H$  as the *faces of  $G$* . The boundary of a face  $f$  is a disjoint set of (not necessarily simple) cycles of  $H$ , which we call *facial walks*. The *length of a facial walk  $W$* , denoted by  $|W|$ , is the number of its edges counted with multiplicity. We remark that a facial walk may consist of only a single vertex, in which case its length is 0. See Figure 6(a) for an example.

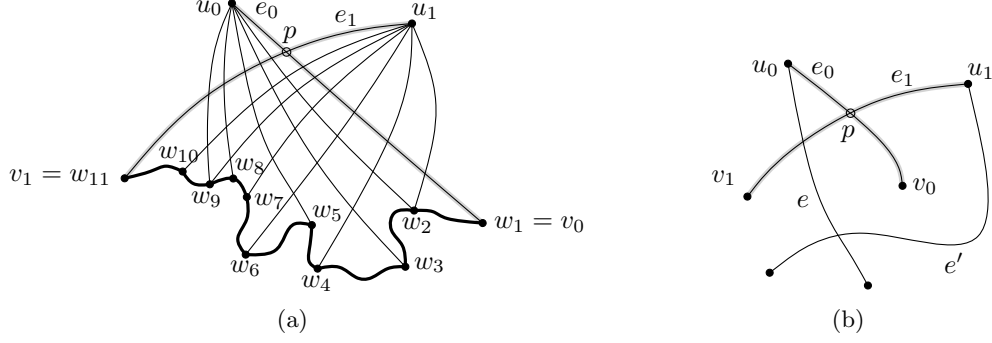


Figure 5: (a) The stars  $S_0$  and  $S_1$  in the proof of Corollary 3 (b) If an edge  $e'$  crosses  $e \in S_0$  between the crossing of  $e$  and  $e_1$  and the endpoint of  $e$  different from  $u_0$ , and  $e' \notin S_1$ , then  $v_0$  is contained in a cell  $c$  bounded by  $e, e'$  and  $e_1$  with  $||c|| = 4$ .

For a face  $f$  and a facial walk  $W$  of  $f$ , we define  $G(W)$  to be the subgraph of  $G$  consisting of the walk  $W$  and all edges that are drawn entirely inside  $f$  and have both endpoints on  $W$ . The set of cells of  $G(W)$  that lie inside  $f$  is denoted by  $C(W)$ . Finally, the graph  $G(W)$  is called a *sunflower* if  $|W| \geq 5$  and  $G(W)$  has exactly  $|W|$  inner edges each of which connects two vertices at distance 2 on  $W$ . See Figure 6(b) for an example of a sunflower. We remark that for convenience we depict facial walks in our figures as simple cycles, even when there are repeated vertices or edges.

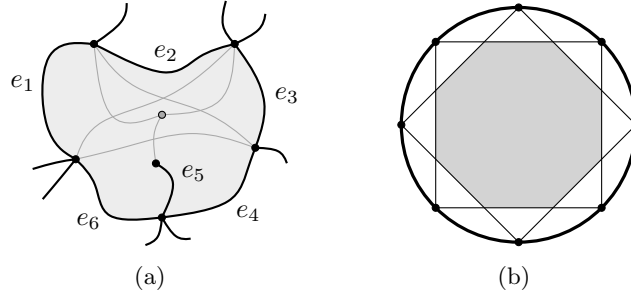


Figure 6: (a) A cell of  $H$  (drawn black) is shown in gray. The boundary of the cell is the cycle  $e_1, e_2, e_3, e_4, e_5, e_5, e_6$ . (b) A sunflower on 8 vertices. The facial walk  $W$  is drawn thick. A cell bounded by 8 edge segments and no vertex is highlighted.

### 3.2 Counting the Number of Edges

We shall count the number of edges of  $G$  in three sets:

- Edges in  $H$ , that is all uncrossed edges.
- Edges in  $E(G(W)) \setminus E(W)$  for every facial walk  $W$ .
- Edges between different facial walks of the same face  $f$  of  $G$ .

The edges in  $H$  will be counted in the final proof of Theorem 1 below. We start by counting the crossed edges, first within the same facial walk and afterwards between different facial walks. For convenience, let us call for a facial walk  $W$  the edges in  $E(G(W)) \setminus E(W)$  and their edge segments *inner edges* and *inner edge segments* of  $G(W)$ , respectively.



**Lemma 2.** *Let  $W$  be any facial walk. If every inner edge segment of  $G(W)$  bounds a cell of  $G(W)$  of size 4 and no cell of  $G(W)$  contains two vertices on its boundary that are not consecutive in  $W$ , then  $G(W)$  is a sunflower.*

*Proof.* Let  $v_0, \dots, v_k$  be the clockwise order of vertices around  $W$ . (In the following, indices are considered modulo  $k+1$ .) For any vertex  $v_i$  we consider the set of inner edges incident to  $v_i$ . Since no two non-consecutive vertices of  $W$  lie on the same cell, every  $v_i$  has at least one such edge. Moreover, note that for each edge  $v_i v_{i+1}$  of  $W$  the unique cell  $c_i$  with  $v_i v_{i+1}$  on its boundary has size at least 5. This implies that every  $v_i$  has indeed at least two incident inner edges. Finally, note that every inner edge is crossed, since otherwise there would be two non-consecutive vertices of  $W$  bounding the same cell of  $G(W)$ .

Now let us consider the clockwise first inner edge incident to  $v_i$ , denoted by  $e_i^1$ . Since an edge segment of  $e_i^1$  bounds the cell  $c_i$  on the other side of this segment is a cell of size 4. This means that  $e_i^1$  and the clockwise next inner edge at  $v_i$  are crossed by some edge  $e$ . By fan-planarity  $e$  crosses only edges incident to  $v_i$ . Thus each endpoint of  $v$  bounds together with  $v_i$  some cell of  $G(W)$ . Since only consecutive vertices of  $W$  bound the same cell of  $G(W)$ , this implies that  $e = v_{i-1} v_{i+1}$ . Since this is true for every  $i \in \{0, \dots, k\}$ , we conclude that  $G(W)$  is a sunflower.  $\square$

Recall that  $C(W)$  denotes the set of all bounded cells of  $G(W)$ .

**Lemma 3.** *For every facial walk  $W$  with  $|W| \geq 3$  we have*

$$|E(G(W)) \setminus E(W)| \leq 2|W| - 5 - \sum_{c \in C(W)} \max\{0, ||c|| - 5\}.$$

*Proof.* Without loss of generality we may assume that  $W$  is a simple cycle. We proceed by induction on  $|E(G(W))|$ . As induction base we consider the case that  $W$  is a triangle. Then  $G(W) = W$  and  $C(W)$  consists of a single cell  $c$  with  $||c|| = 6$ . Thus

$$|E(G(W)) \setminus E(W)| = 0 = 2|W| - 5 - (||c|| - 5).$$

First, consider any inner edge segment  $e^*$  and the two cells  $c_1, c_2 \in C(W)$  containing  $e^*$  on their boundary. If  $c^*$  denotes the set  $c_1 \cup c_2$  in  $G(W) \setminus e$ , then

$$||c^*|| = ||c_1|| + ||c_2|| - 4$$

and thus

$$\max\{0, ||c^*|| - 5\} = \max\{0, ||c_1|| - 5\} + \max\{0, ||c_2|| - 5\} + x, \quad (1)$$

where  $x = 1$  if  $||c_1|| \geq 5$  and  $||c_2|| \geq 5$  and  $x = 0$  otherwise.

Now, we shall distinguish three cases:  $G(W)$  is a sunflower, some inner edge segment is not bounded by a cell of size 4, and some cell of  $G(W)$  contains two vertices on its boundary that are not consecutive in  $W$ . By Lemma 2 this is a complete case distinction.

*Case 1.*  $G(W)$  is a sunflower. Then by definition,  $G(W)$  has exactly  $|W|$  inner edges. Moreover,  $C(W)$  contains exactly one cell  $c$  of size greater than 4 and for that cell we have  $||c|| = |W|$ . Thus

$$|E(G(W)) \setminus E(W)| = |W| = 2|W| - 5 - (|W| - 5).$$

Case 2. Some edge segment  $e^*$  of some inner edge  $e$  bounds two cells  $c_1, c_2$  of size at least 5 each. Then applying induction to the graph  $G' = G(W) \setminus e$  we get

$$\begin{aligned} |E(G(W)) \setminus E(W)| &= 1 + |E(G') \setminus E(W)| \leq 1 + 2|W| - 5 - \sum_{c \in C(G')} \max\{0, \|c\| - 5\} \\ &\stackrel{(1)}{=} 1 + 2|W| - 5 - \sum_{c \in C(W)} \max\{0, \|c\| - 5\} - 1. \end{aligned}$$

Case 3. Some cell of  $G(W)$  contains two vertices  $u, w$  on its boundary that are not consecutive on  $W$ . Note that  $uw$  may or may not be an inner edge of  $G(W)$ . In the latter case we denote by  $c^*$  the unique cell that is bounded by  $u$  and  $w$ . In any case exactly two cells  $c_1, c_2$  of  $G(W) \cup uw$  are bounded by  $u$  and  $w$  and we have  $\|c^*\| = \|c_1\| + \|c_2\| - 4$ , provided  $c^*$  exists.

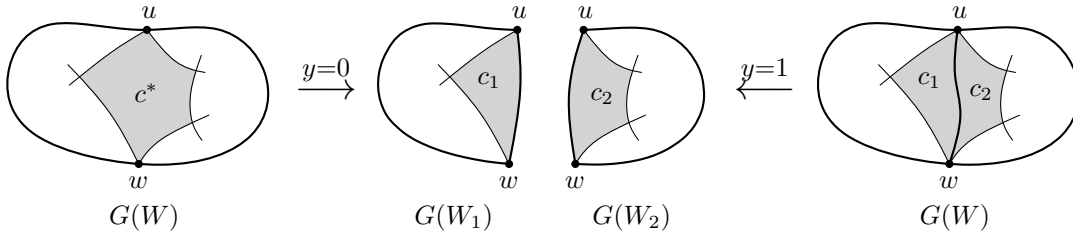


Figure 7: The graph  $G(W)$  is split into two graphs  $G(W_1)$  and  $G(W_2)$  along two vertices  $u, w$  that are not consecutive on  $W$  but bound the same cell of  $G(W)$ .

We consider the two cycles  $W_1, W_2$  in  $W \cup uw$  that are different from  $W$ , such that  $W_1$  surrounds  $c_1$  and  $W_2$  surrounds  $c_2$ . For  $i = 1, 2$  consider  $G(W_i)$ , i.e., the subgraph of  $G(W) \cup uw$  induced by  $W_i$ , see Figure 7. We have

$$\begin{aligned} |W| &= |W_1| + |W_2| - 2, \\ |E(G(W)) \setminus E(W)| &= |(E(G(W_1)) \setminus E(W_1))| + |(E(G(W_2)) \setminus E(W_2))| + y, \\ \sum_{c \in C(W)} \max\{0, \|c\| - 5\} &\stackrel{(1)}{=} \sum_{c \in C(W_1)} \max\{0, \|c\| - 5\} + \sum_{c \in C(W_2)} \max\{0, \|c\| - 5\} + (1 - y), \end{aligned}$$

where  $y = 1$  if  $uw$  already was an inner edge of  $G(W)$  and  $y = 0$  otherwise. Now, applying induction to  $G(W_1)$  and  $G(W_2)$  gives the claimed bound. □

Let us define by  $C(f)$  the union of  $C(W)$  for all facial walks  $W$  of  $f$ . Moreover, we partition  $C(f)$  into  $C_\emptyset(f)$  and  $C_*(f)$ , where a cell  $c \in C(f)$  lies in  $C_\emptyset$  if and only if  $(c \setminus \partial c) \cap V(G) = \emptyset$ . I.e., cells in  $C_\emptyset(f)$  do not have any vertex of  $G$  in their open interior, whereas cells in  $C_*(f)$  contain some vertex of  $G$  in their interior. Without loss of generality we have that for each  $C_*(f)$  is either empty or contains at least one bounded cell. This can be achieved by picking a cell of  $G$  that has the maximum number of surrounding Jordan curves of the form  $\partial c$  for  $c \in \bigcup_f C_*(f)$ , and defining it to be in the unbounded cell of  $G$ .

Before we bound the number of edges between different facial walks of  $f$  we need one more lemma. Consider a face  $f$  of  $G$  with at least two facial walks and a cell  $c \in C_*(f)$  that is inclusion-minimal. Let  $W_1$  be the facial walk with  $c \in C(W_1)$  and  $W_2, \dots, W_k$  be the facial walks that are contained in  $c$ . For  $i = 1, \dots, k$  let  $c_i$  be the cell of  $G(W_i)$  that contains all walks  $W_j$  with  $j \neq i$ .

In particular, we have  $c_1 = c$ . Moreover, we call an edge between two distinct facial walks  $W_i$  and  $W_j$  a  $W_iW_j$ -edge.

**Lemma 4.** *Exactly one of  $c_1, \dots, c_k$  has a vertex on its boundary.*

*Proof.* We proceed by proving a series of claims first.

**Claim 1.** *If a  $W_iW_j$ -edge and a  $W_{i'}W_{j'}$ -edge cross, then  $\{i, j\} = \{i', j'\}$ .*

*Proof of Claim.* Consider a  $W_iW_j$ -edge  $e_0 = u_0v_0$  crossing a  $W_{i'}W_{j'}$ -edge  $e_1 = u_1v_1$ . By Corollary 1 one endpoint of  $e_0$ , say  $u_0 \in W_i$ , and one endpoint of  $e_1$ , say  $u_1 \in W_{i'}$ , are joint by an uncrossed edge. In particular,  $W_i = W_{i'}$ .

If, Case 1,  $e_0$  is crossed by a second edge incident to  $v_1$ , then applying Lemma 1 gives an uncrossed edge  $u_0v_1$ , which is a contradiction to the fact that  $W_{j'} \neq W_{i'}$ , or an uncrossed edge  $v_0v_1$ , which implies  $W_j = W_{j'}$  as desired.

Otherwise, Case 2,  $e_0$  is crossed only by edges at  $u_1$ , and by symmetry  $e_1$  is crossed only by edges at  $u_0$ . Applying Corollary 3 we get that  $v_0$  and  $v_1$  are in the same connected component of  $H$  and hence  $W_j = W_{j'}$ , as desired.  $\triangle$

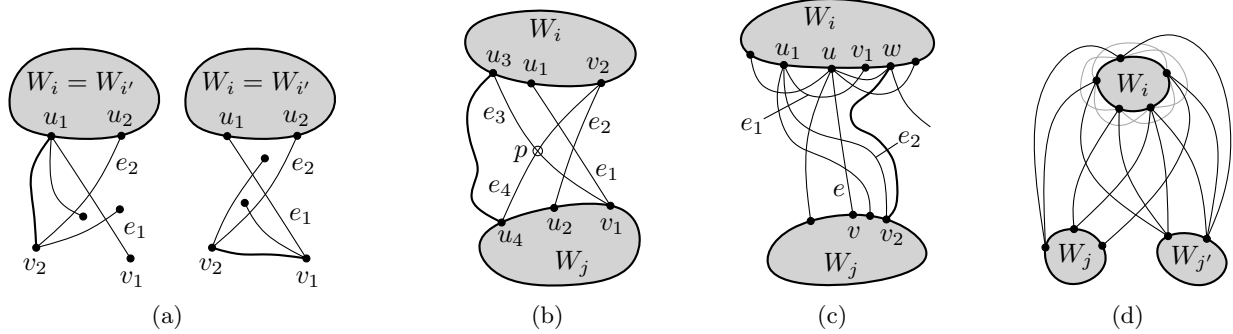


Figure 8: (a) Case 1 in the proof of Claim 1. Illustrations of the proofs of Claim 2 (b), Claim 3 (c) and Claim 4 (d).

For a facial walk  $W_i$  a vertex  $v \in W_i$  is called *open* if  $v$  lies on  $\partial c_i$ . Moreover, a vertex  $v \in W_i$  is called *closed* if  $v$  is not open but there is at least one edge between  $v$  and another facial walk  $W_j \neq W_i$ . So every edge between distinct facial walks has endpoints that are open or closed, and by fan-planarity at least one endpoint is open.

**Claim 2.** *If two  $W_iW_j$ -edges cross then both have exactly one open end, which moreover are in the same facial walk.*

*Proof of Claim.* Let  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  be two crossing  $W_iW_j$ -edges. Assume for the sake of contradiction that  $e_1$  has an open endpoint  $u_1 \in W_i$  and  $e_2$  has an open endpoint  $u_2 \in W_j$ . We consider the edges  $e_3 = u_3v_1$  and  $e_4 = u_4v_2$  that are incident to  $v_1$  and  $v_2$  respectively, cross each other and whose crossing point  $p$  is furthest away from  $v_1$  and  $v_2$ . See Figure 8(b) for an illustration. Note that possibly  $e_1 = e_3$  and/or  $e_4 = e_2$ .

Now  $u_3$  is not in  $W_j$  because  $u_1$  is an open endpoint and  $u_4$  is not in  $W_i$  because  $u_2$  is an open end. Hence by Claim 1  $u_3 \in W_i$  and  $u_4 \in W_j$ . Moreover, by Lemma 1  $u_3u_4$  is an uncrossed edge of  $G$  – a contradiction to the fact that  $W_i$  and  $W_j$  are distinct facial walks.  $\triangle$

Claim 2 implies that every edge between different facial walks has exactly one open endpoint and one closed end, because every such edge with two open endpoints would be crossed by some other edge between two facial walks.

**Claim 3.** *If a  $W_iW_j$ -edge has a closed endpoint  $u \in W_i$  and  $w$  is the counterclockwise next open or closed vertex of  $W_i$  after  $u$ , then there exists a  $W_iW_j$ -edge incident to  $w$  with open endpoint in  $W_j$ .*

*Proof of Claim.* Let  $e = uv$  be a  $W_iW_j$ -edge that has a closed endpoint  $u \in W_i$ . By fan-planarity  $v$  is an open vertex of  $W_j$ . Consider the crossing of  $e$  closest to  $u$  and let  $e_1 = u_1v_1$  be the crossing edge. Clearly,  $e_1$  is an edge from  $G(W_i)$ , where without loss of generality  $v_1$  comes counterclockwise after  $u$  in  $W_i$ . Further assume without loss of generality that  $e$  is the  $W_iW_j$ -edge at  $u$  whose crossing with  $e_1$  is closest to  $v_1$ . If  $e$  is not crossed between  $v$  and its crossing with  $e_1$  then we can draw a  $W_iW_j$ -edge between  $v$  and  $w$  that is not crossed by any edge between facial walks and we are done.

Otherwise, if  $e$  is crossed by some edge  $e_2$  between its crossing with  $e_1$  and  $v$ , then by fan-planarity  $e_2$  is incident to  $u_1$  or  $v_1$ . Moreover, by Claim 1 and Claim 2  $e_2$  has a closed endpoint in  $W_i$  and an open endpoint in  $W_j$ . Thus if  $e_2$  is incident to  $v_1$ , then we have found the desired  $W_iW_j$ -edge. So assume that  $e_2 = u_1v_2$  for some  $v_2 \in W_j$ . Moreover, let  $e_2$  be the  $W_iW_j$ -edge whose crossing with  $e$  is closest to  $u$ . We refer to Figure 8(c) for an illustration.  $\triangle$

Because  $e_2$  has a closed endpoint  $u_1 \in W_i$  every edge crossing  $e_1$  or  $e_2$  endpoints in  $u$ . Thus by the choice of  $e$  we conclude that  $e_2$  is not crossed between  $v_2$  and its crossing with  $e$  and that  $e_1$  is not crossed between its crossing with  $e$  and the next vertex or edge in  $G(W_i)$ . Moreover, by the choice of  $e_2$  the edge  $e$  is not crossed between its crossings with  $e_2$  and  $e_1$ . Thus we can draw a  $W_iW_j$ -edge from  $v_2$  to  $w$ .  $\triangle$

Claim 3 together with Claim 2 implies that on each facial walk every closed vertex is followed by another closed vertex. In particular, the facial walks come in two kinds, one with open vertices only and one with closed vertices only. We remark one can show that, if  $W_i$  has only closed vertices, then  $G(W_i)$  is a sunflower.

**Claim 4.** *Every facial walk with only closed vertices has edges to exactly one facial walk with only open vertices.*

*Proof of Claim.* Assume for the sake of contradiction that facial walk  $W_i$  with only closed vertices has edges to two different facial walks  $W_j, W_{j'}$  with only open vertices. Claim 3 implies that if some closed vertex of  $W_i$  has an edge to  $W_j$ , then every closed vertex of  $W_i$  has an edge to  $W_j$ , and the same is true for  $W_{j'}$ . Hence, each of the at least three closed vertices in  $W_i$  has edge to  $W_j$  and  $W_{j'}$ , which implies that some  $W_iW_j$ -edge and some  $W_iW_{j'}$ -edge must cross, see Figure 8(d). (Indeed, if any two such edges would not cross, then contracting  $W_j$  and  $W_{j'}$  into a single point each and placing a new vertex in the middle of  $W_i$  with an edge to every closed vertex in  $W_i$  would give a planar drawing of  $K_{3,3}$ .) Thus by Claim 1 we have  $W_j = W_{j'}$  – a contradiction to our assumption.  $\triangle$

We are now ready to prove that at most one facial walk has open vertices. Recall that by Claim 3 every facial walk is of one of two kinds: only open vertices or only closed vertices. Moreover, by fan-planarity and Claim 2 no edge runs between two facial walks of the same kind. We consider a bipartite graph  $F$  whose black and white vertices correspond to facial walks of the first and second kind, respectively, and whose edges correspond to pairs  $W_i, W_j$  of facial walks for which there is at least one  $W_iW_j$ -edge in  $G$ . Since  $G$  is connected,  $F$  is connected, and by Claim 4 every white vertex is adjacent to exactly one black vertex. This means that  $F$  is a star and has exactly one black vertex, which concludes the proof.  $\square$

Having Lemma 4 we can now bound the number of  $W_i W_j$ -edges. Recall that  $W_1, \dots, W_k$  denote the facial walks for the fixed face  $f$  of  $G$ , and that for  $i = 1, \dots, k$  we denote by  $c_i$  the cell of  $G(W_i)$  containing all  $W_j$  with  $j \neq i$ .

**Lemma 5.** *The number of edges between  $W_1, \dots, W_k$  is at most*

$$4(k-2) + \sum_{i=1}^k \|c_i\|.$$

*Proof.* By Lemma 4 exactly one of  $c_1, \dots, c_k$  has vertices on its boundary, say  $W_1$ . Let  $U$  be the set of vertices on the boundary of  $c_1$ . For a vertex  $u \in U$  and an index  $i \in \{2, \dots, k\}$  we call an edge between  $u$  and  $W_i$  a  $uW_i$ -edge. We define a bipartite graph  $J$  as follows. One bipartition class is formed by the vertices in  $U$ . In the second bipartition class there is one vertex  $w_i$  for each facial walk  $W_i$ ,  $i = 1, \dots, k$ . A vertex  $u \in U$  is connected by an edge to  $w_i$  if and only if  $i = 1$  or  $i \geq 2$  and there is a  $uW_i$ -edge.

**Claim 5.** *The graph  $J$  is planar.*

*Proof of Claim.* We consider the following embedding of  $J$ . Afterwards we shall argue that this embedding is indeed a plane embedding. So take the position of every vertex  $u \in U$  from the fan-planar embedding of  $G$ . For  $i \geq 2$ , we consider the drawing of  $W_i$  in the embedding of  $G$ , for each edge between a vertex  $u \in U$  and the vertex  $w_i$  in  $J$  we take the drawing of one  $uW_i$ -edge in  $G$ , and then contract the drawing of  $W_i$  into a single point – the position for vertex  $w_i$ . Finally, we place the last vertex  $w_1$  outside the cell  $c_1$  and connect  $w_1$  to each  $u \in U$  in such a way that these edges do not cross any other edge in  $J$ . See Figure 9(a) for an illustrating example.

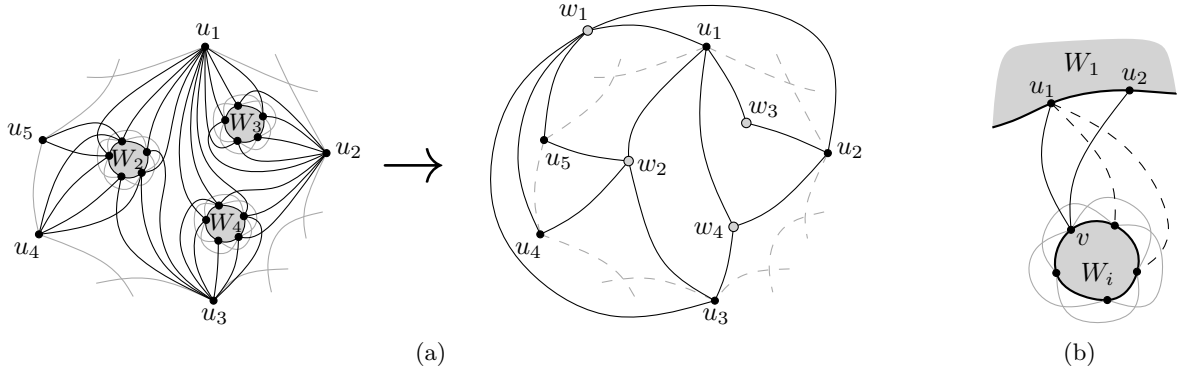


Figure 9: (a) Obtaining the graph  $J$ . (b) The contradiction in Claim 6.

Now the resulting drawing of  $J$  contains crossing edges only if a  $uW_i$ -edge crosses a  $u'W_{i'}$ -edge in  $G$ . However, by Lemma 4 the cells  $c_2, \dots, c_k$  have no vertices on their boundary. Consequently, for each  $i = 2, \dots, k$  every  $uW_i$ -edge crosses an edge of  $G(W_i)$ . Now if a  $uW_i$ -edge  $e$  would cross a  $u'W_{i'}$ -edge with  $u \neq u'$  and  $i \neq i'$ , then  $e$  would be crossed by two independent edges, contradicting the fan-planarity of  $G$ .  $\triangle$

Since  $J$  is a planar bipartite graph with bipartition classes of size  $|U|$  and  $k$  we have

$$|E(J)| = \sum_{i=1}^k \deg_J(w_i) \leq 2(|U| + k) - 4.$$

**Claim 6.** For each  $i = 2, \dots, k$  the number of  $uW_i$ -edges is at most

$$||c_i|| + 2 \deg_J(w_i).$$

*Proof of Claim.* Consider the vertices on  $W_i$  and the set  $U' \subseteq U$  of vertices on  $W_1$  that have a neighbor on  $W_i$ . For each  $u \in U'$  the  $uW_i$ -edges form a consecutive set in the cyclic ordering of edges around  $u$ . Since not every edge at  $u$  is a  $uW_i$ -edge (at least one edge endpoints in  $W_1$ ) we obtain a linear order on the  $uW_i$ -edges going counterclockwise around  $u$ .

Now we claim that when we remove for each  $u \in U'$  the last two  $uW_i$ -edges in the linear order for  $u$ , then every vertex  $v$  in  $W_i$  is the endpoint of at most one  $uW_i$ -edge. Indeed, if after the edges have been removed two vertices  $u_1, u_2 \in U'$  have a common neighbor  $v$  on  $W_i$ , then at least two  $u_1W_i$ -edges cross the edge  $u_2v$  (or the other way around). However, not both these edges can endpoint at the same vertex on  $W_i$ , and thus  $u_2v$  is crossed by two independent edges, one  $u_1W_i$ -edge and one edge in  $G(W_i)$  – a contradiction to the fan-planarity of  $G$ . So the number of  $uW_i$ -edges is at most  $2|U'| + |W_i| = ||c_i|| + 2 \deg_J(w_i)$ .  $\triangle$

We can now bound the total number of  $uW_i$ -edges with  $i \geq 2$  as follows.

$$\begin{aligned} \sum_{i=2}^k \#uW_i\text{-edges} &\leq \sum_{i=2}^k (||c_i|| + 2 \deg_J(w_i)) \\ &= \sum_{i=2}^k ||c_i|| + 2|E(J)| - 2 \deg_J(w_1) \\ &\leq \sum_{i=2}^k ||c_i|| + 4(|U| + k) - 8 - 2|U| \\ &= \sum_{i=2}^k ||c_i|| + 2|U| + 4(k - 2) \leq \sum_{i=2}^k ||c_i|| + ||c_1|| + 4(k - 2) \end{aligned}$$

$\square$

We continue by bounding the total number of crossed edges of  $G$  that are drawn inside a fixed face  $f$  of  $G$ . To this end let  $k_f$  be the number of distinct facial walks of  $f$  and  $|f|$  be the sum of lengths of facial walks of  $f$ , i.e.,  $|f| = \sum_{W \text{ facial walk of } f} |W|$ .

**Lemma 6.** The number of edges inside  $f$  is at most

$$2|f| + 5(k_f - 2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\}.$$

*Proof.* We do induction on  $k_f$ .

First let  $k_f = 1$ , i.e., the face  $f$  is bounded by a unique facial walk  $W$ . Then by Lemma 3 there are at most  $2|W| - 5 - \sum_{c \in C(W)} \max\{0, ||c|| - 5\}$  edges inside  $f$ . With  $|W| = |f|$  and  $C_\emptyset(f) = C(W)$  this gives the claimed bound.

Now assume that  $k_f \geq 2$ , i.e., the face  $f$  has  $k = k_f$  distinct facial walks  $W_1, \dots, W_k$ . Let  $c_1$  be an inclusion-minimal cell in  $(C(W_1) \cup \dots \cup C(W_k)) \setminus C_\emptyset(f)$ . Without loss of generality let  $W_1$  be the facial walk with  $c_1 \in C(W_1)$  and  $W_2, \dots, W_j$  be the facial walks of  $f$  that lie inside  $c_1$ . In particular we have  $2 \leq j \leq k$ . Let  $G'$  be the graph that is obtained from  $G$  after removing all

vertices that lie inside  $c_1$ . We consider  $G'$  with its fan-planar embedding inherited from  $G$ . Clearly, the face  $f'$  in  $G'$  corresponding to  $f$  in  $G$  has exactly  $k - (j - 1) < k$  facial walks and we have

$$|f| = |f'| + |W_2| + \cdots + |W_j|.$$

For  $i = 2, \dots, j$  we denote by  $c_i$  the cell of  $G(W_i)$  containing  $W_1$ . Moreover, let  $C = C(W_2) \cup \cdots \cup C(W_j)$ . Then

$$C_\emptyset(f) = (C_\emptyset(f') \cup C) \setminus \{c_1, c_2, \dots, c_j\}.$$

Further we partition the edges inside  $f$  into three disjoint sets  $E_1, E_2, E_3$  as follows:

- The edges in  $E_1$  are precisely the edges of  $G'$  inside  $f'$ .
- The edges in  $E_2$  are precisely the edges of  $G$  between  $W_1$  and  $W_2 \cup \cdots \cup W_j$ .
- $E_3 = (E(G(W_2)) \setminus E(W_2)) \cup \cdots \cup (E(G(W_j)) \setminus E(W_j))$ .

Now by induction hypothesis we have

$$|E_1| \leq 2|f'| + 5(k - j - 1) - \sum_{c \in C_\emptyset(f')} \max\{0, ||c|| - 5\}.$$

By Lemma 5 we have

$$|E_2| \leq \sum_{i=1}^j ||c_i|| + 4(j - 2) \leq \sum_{i=1}^j \max\{0, ||c_i|| - 5\} + 9j - 8.$$

By Lemma 3 we have

$$|E_3| \leq 2(|W_2| + \cdots + |W_j|) - 5(j - 1) - \sum_{c \in C} \max\{0, ||c|| - 5\}.$$

Plugging everything together we conclude that the number of edges of  $G$  inside  $f$  is at most

$$\begin{aligned} |E_1 \dot{\cup} E_2 \dot{\cup} E_3| &\leq 2|f'| + 5(k - j - 1) - \sum_{c \in C_\emptyset(f')} \max\{0, ||c|| - 5\} \\ &\quad + \sum_{i=1}^j \max\{0, ||c_i|| - 5\} + 9j - 8 \\ &\quad + 2(|W_2| + \cdots + |W_j|) - 5(j - 1) - \sum_{c \in C} \max\{0, ||c|| - 5\} \\ &= 2|f| + 5(k - 2) - (j - 2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\} \\ &\leq 2|f| + 5(k_f - 2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\}, \end{aligned}$$

which concludes the proof.  $\square$

Note that Lemma 6 implies that inside a face  $f$  of  $H$  there are at most  $2|f| + 5(k_f - 2)$  edges. Having this, we are now ready to prove our main theorem. For convenience we restate it here.

**Theorem 1.** *Every simple topological graph  $G$  on  $n \geq 3$  vertices with neither configuration I nor configuration II has at most  $5n - 10$  edges. This bound is tight for  $n \geq 20$ .*

*Proof.* Consider a fan-planar graph  $G = (V, E)$  on  $n$  vertices with properties (i) and (ii). Let  $H$  be the spanning subgraph of  $G$  on all uncrossed edges. In particular

$$V(H) = V(G).$$

Let us denote by  $F(H)$  the set of all faces of  $H$ . Since every edge  $e \in E(H)$  appears either exactly once in two distinct facial walks or exactly twice in the same facial walk, we have

$$\sum_{f \in F(H)} |f| = 2|E(H)|.$$

Further we denote by  $k_f$  the number of facial walks for a given face  $f$ , and by  $CC(H)$  the number of connected components of  $H$ . Since a face with  $k$  facial walks gives rise to  $k$  connected components of  $H$ , we have

$$\sum_{f \in F(H)} (k_f - 1) = CC(H) - 1.$$

Hence we conclude

$$\begin{aligned} |E(G)| &\stackrel{\text{Lemma 6}}{\leq} |E(H)| + \sum_{f \in F(H)} (2|f| + 5(k_f - 2)) \\ &= |E(H)| + 2 \sum_{f \in F(H)} |f| + 5 \sum_{f \in F(H)} (k_f - 1) - 5|F(H)| \\ &= 5|E(H)| + 5CC(H) - 5|F(H)| - 5 = 5|V(H)| - 10, \end{aligned}$$

where the last equation is Euler's formula for the plane embedded graph  $H$ . With  $|V(H)| = |V(G)| = n$  this concludes the proof.  $\square$

## 4 Discussion

We have shown that every simple  $n$ -vertex graph without configurations I and II has at most  $5n - 10$  edges. Of course, if we allow  $G$  to have parallel edges or loops, there could be arbitrarily many edges, even if the drawing of  $G$  is planar. However, if we allow only *non-homeomorphic parallel edges* and only *non-trivial loops*, then  $G$  has a bounded number of edges. Here, two parallel edges are non-homeomorphic and a loop is non-trivial if the bounded component of the plane after the removal of both parallel edges, respectively the loop, contains at least one vertex of  $G$ . Note for instance that Euler's formula still holds for plane graphs with non-homeomorphic parallel edges and non-trivial loops, and that in this case every face still has length at least 3. Therefore any such plane graph with  $n$  vertices still has at most  $3n - 6$  edges. We strongly conjecture that our  $5n - 10$  bound also holds if non-homeomorphic parallel edges and non-trivial loops are allowed.

Another relaxation would be to allow non-simple topological graphs, i.e., to allow edges to cross more than once and incident edges to cross. It would be interesting to see whether there is an  $n$ -vertex non-simple fan-planar graph with strictly more than  $5n - 10$  edges. However, let us remark that if we allow both, non-simple drawings and non-homeomorphic parallel edges, then there are 3-vertex topological graph with arbitrarily many edges. Let us simply refer to Figure 10(a) for such an example. The idea is to start with an edge  $e_1$  from  $u$  to  $v$ , and edge  $e_i$  starts clockwise next to  $e_{i-1}$  at  $u$  goes in parallel with  $e_{i-1}$  until  $e_{i-1}$  endpoints at  $v$ , where  $e_i$  goes a little further surrounding vertex  $w$  once and then ending at  $v$ . This way no two such parallel edges are homeomorphic.



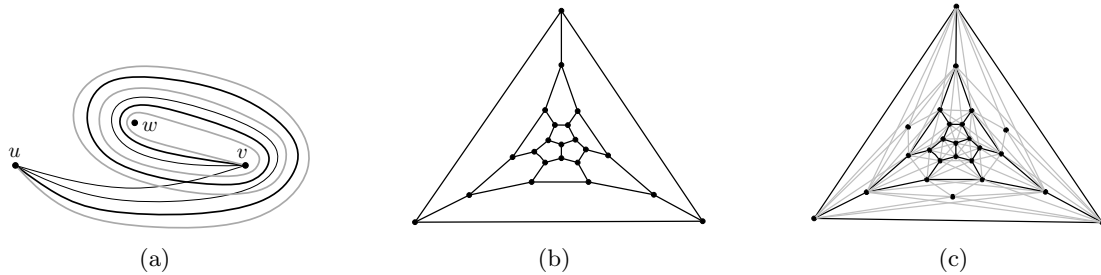


Figure 10: (a) A topological non-simple fan-planar graph with arbitrarily many edges. (b) The modified dodecahedral graph without the extensions and (c) fully extended to obtain  $5n - 11$  straight-line edges.

Also, one can relax fan-planarity to *k-fan-planarity* for some  $k \geq 1$ , where every edge may only be crossed by  $k$  fan-crossings. We remark that a simple probabilistic argument shows that for fixed  $k$  every  $n$ -vertex  $k$ -fan-planar graph has only linearly many edges, see Lemma 2.9 in [2]. However, exact bounds are not known.

It can also be interesting to consider strengthenings of fan-planar graphs, e.g., to consider straight-line fan-planar embeddings. Note that the dodecahedral graph with pentagrams which was a tight example of the  $5n - 10$  bound, can be extended as follows to obtain a straight-line fan-planar graph with  $5n - 11$  edges: Replace one vertex of the dodecahedron by a single triangle, which is used as the outer face. Draw the planar graph with convex faces such that all (additional) edges can be drawn straightline without producing unnecessary crossings, cf. Figure 10(b). The 3 adjacent pentagons now converted to hexagons are extended by 2-hops and spokes as explained in Proposition 1, i.e., by one additional vertex and 12 edges each. We do not suspect that an  $n$ -vertex straight-line fan-planar graph can have  $5n - 10$  edges.

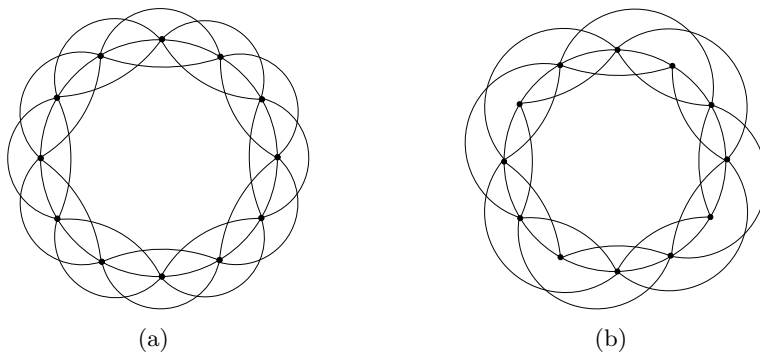


Figure 11: (a) An edge-maximal fan-planar graph with non-homeomorphic parallel edges on  $3n$  edges. (b) An edge-maximal simple fan-planar graph on  $\frac{8}{3}n$  edges.

Finally, one is usually also interested in edge-maximal topological graphs with as *few* edges as possible. In our case we can construct edge-maximal fan-planar graphs on no more than  $3n$  edges if parallel edges are allowed (Figure 11(a)) and no more than  $\frac{8}{3}n$  edges if parallel edges are not allowed (Figure 11(b)). We suspect these examples to be best-possible.

Let us summarize some possible research directions.

**Problems.** *Each of the following is open.*

- P1:** What is the maximum number of edges in a simple topological graph  $G$  with forbidden configuration I, but where configuration II is allowed?
- P2:** Is there an  $n$ -vertex simple fan-planar graph with non-homeomorphic parallel edges and/or non-trivial loops with strictly more than  $5n - 10$  edges?
- P3:** Does the  $5n - 10$  upper bound also hold for non-simple fan-planar graphs?
- P4:** For  $k \geq 2$  what is the largest number of edges in an  $n$ -vertex  $k$ -fan-planar graph?
- P5:** Prove that the  $5n - 11$  bound is tight for straight-line fan-planar embeddings similar to the  $4n - 9$  bound for straight-line embedded 1-planar graphs [12].
- P6:** How many edges has an  $n$ -vertex edge-maximal graph without configurations I and II at least?

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